Lesson 4: Limits, Part 3 (Evaluating Limits Analytically)

If $f(x)$ is a:

- polynomial fun. ($3x^9 - 7x^2$)
- rational function ($\frac{\text{poly.}}{\text{poly.}}$ $\frac{x^2+2x-3}{2x^2-18}$)
- radical function ($\sqrt[3]{x}$)
- trig function
- exponential function (e^x, 3^x)
- log function ($\ln x$, $\log_3 x$)

or a combination of these types of functions **and** $f(x)$ is defined at $x=c$, we can find $\lim\limits_{x \to c} f(x)$ by direct substitution.

Ex 1 (a) $\lim\limits_{x \to 2} 4x-9 = 4(2)-9 = \boxed{-1}$

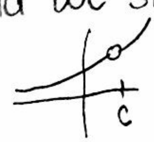
(b) $\lim\limits_{x \to 3} \sin(\ln(e^x)) = \sin(\ln(e^3))$

$= \boxed{\sin(3)}$ ← don't round!

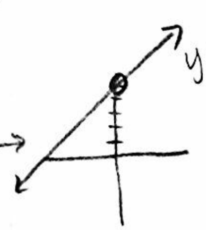
If $f(x)$ isn't defined, (in this class) either $f(c) = \frac{0}{0}$ or $\frac{a}{0}$ ← $\neq 0$

- Case 1: $f(c) = \frac{0}{0}$

  Then $f(x)$ has a hole at $x=c$, and we should be able to factor and cancel.



Ex 2 $\lim\limits_{x \to 0} \frac{\cancel{x}(x+5)}{\cancel{x}} = \lim\limits_{x \to 0} x+5 = 0+5 = \boxed{5}$ →

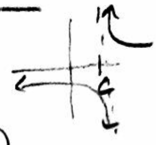


Ex 3 $\lim\limits_{x \to -1} \frac{x^2-1}{x^2+3x+2} = \lim\limits_{x \to -1} \frac{\cancel{(x+1)}(x-1)}{(x+2)\cancel{(x+1)}}$

$= \frac{0}{0}$

$= \frac{-1-1}{-1+2} = \frac{-2}{1} = \boxed{-2}$

Case 2: $f(c) = \frac{a \neq 0}{0}$, Then $f(x)$ has a vertical asymptote at $x=c$.



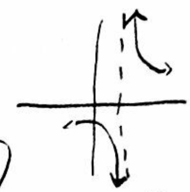
Ex 4 $\lim_{x\to 3} \frac{1}{(x-3)^2} = \frac{1}{0}$

- $\lim_{x\to 3^+} \frac{1}{(x-3)^2} = \frac{1}{(+\text{SMALL})^2} = \frac{1}{+\text{SMALL}} = \infty$

  $\left(\frac{1}{\text{SMALL}} = \text{LARGE}\right)$

- $\lim_{x\to 3^-} \frac{1}{(x-3)^2} = \frac{1}{(-\text{SMALL})^2} = \frac{1}{+\text{SMALL}} = \infty$

$\lim_{x\to 3} \frac{1}{(x-3)^2} = \boxed{\infty}$



Ex 5 $\lim_{x\to 1} \frac{3x}{x^2+x-2} = \frac{3}{0}$

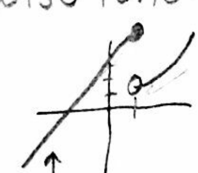
- $\lim_{x\to 1^+} \frac{3x}{(x+2)(x-1)} = \frac{3}{+3(+\text{SMALL})} = +\infty$
- $\lim_{x\to 1^-} \frac{3x}{(x+2)(x-1)} = \frac{3}{+3(-\text{SMALL})} = -\infty$

$\lim_{x\to 1} \frac{3x}{x^2+x-2}$ $\boxed{\text{DNE}}$

Ex 6 $\lim_{x\to 1} \frac{-2}{x-1}$ (DNE)

Piecewise Functions

Use the appropriate part of the piecewise function.



Ex 7 $f(x) = \begin{cases} x+3 & x \leq 1 \\ e^{x-1} & x > 1 \end{cases}$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x+3 = \boxed{3}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x+3 = 1+3 = \boxed{4}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} e^{x-1} = e^0 = \boxed{1}$

$\lim_{x\to 1} f(x) = \boxed{\text{DNE}}$

Ex 8 $f(x) = \begin{cases} x-1, & x \le 2 \\ x^2-2x+1, & x > 2 \end{cases}$

$\lim_{x \to 2} f(x) = ?$

$$\left.\begin{array}{ll} \lim_{x \to 2^-} f(x) = 2-1 = 1 & (x-1) \\ \lim_{x \to 2^+} f(x) = 4-4+1 = 1 & (x^2-2x+1) \end{array}\right\} \lim_{x \to 2} f(x) = \boxed{1}$$

Properties of Limits

- $\lim_{x \to c} af(x) = a \lim_{x \to c} f(x)$ (can pull constants out)
- $\lim_{x \to c} (f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- $\lim_{x \to c} f(x)g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$
- $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ ← if this $\neq 0$